Analytic Solutions for Coupled Vector Volterra Tests

Setup: The Similarity Transform Method

All three examples are constructed by the same technique. Start with a *diagonal* system whose solution is known in closed form, then conjugate by an invertible matrix P to obtain a *fully coupled* system with a known solution.

We use

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad P^{-1} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

One can verify $PP^{-1} = I$ directly. For any diagonal matrix $D = \mathrm{diag}(d_1, d_2)$,

$$\widetilde{D} \;:=\; PDP^{-1} \;=\; \begin{pmatrix} \frac{d_1+d_2}{2} & \frac{d_1-d_2}{2} \\ \frac{d_1-d_2}{2} & \frac{d_1+d_2}{2} \end{pmatrix}.$$

In every example below we choose $d_1 = 1$, $d_2 = 2$, giving

$$\widetilde{D} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

If $\mathbf{y}(t)$ solves the diagonal system, the transformed solution is $\mathbf{z}(t) = P\mathbf{y}(t)$, and all coefficient matrices/functions transform as $(\,\cdot\,) \mapsto P(\,\cdot\,)P^{-1}$, while forcing functions transform as $\mathbf{g} \mapsto P\mathbf{g}$.

1 Type-1 Volterra Integral Equation (VIE-1)

The vector VIE-1 is

$$\mathbf{g}(t) = \int_0^t \widetilde{K}(t-s)\,\mathbf{z}(s)\,ds, \qquad t \geq 0.$$

Diagonal system

Choose the diagonal kernel $K = \mathrm{diag}(1, 2)$ (constant) and the exact solutions

$$y_1(t) = 1 + t, \qquad y_2(t) = t.$$

The corresponding forcing functions are obtained by direct integration:

$$g_1(t) = \int_0^t 1 \cdot (1+s)\,ds = t + \tfrac{t^2}{2},$$

$$g_2(t) = \int_0^t 2s\,ds = t^2.$$

Transformed system

Applying P:

$$\mathbf{z}(t) = P\mathbf{y}(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1+t \\ t \end{pmatrix} = \begin{pmatrix} 1+2t \\ 1 \end{pmatrix},$$

$$\widetilde{K} = PDP^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix},$$

$$\widetilde{\mathbf{g}}(t) = P\mathbf{g}(t) = \begin{pmatrix} (t+\frac{t^2}{2}) + t^2 \\ (t+\frac{t^2}{2}) - t^2 \end{pmatrix} = \begin{pmatrix} t + \frac{3t^2}{2} \\ t - \frac{t^2}{2} \end{pmatrix}.$$

Verification

We check that $\widetilde{\mathbf{g}}(t) = \int_0^t \widetilde{K}(t-s)\,\mathbf{z}(s)\,ds$ component by component.

Component 0.

$$\begin{aligned} \int_0^t \Big(\tfrac{3}{2}z_0(s) - \tfrac{1}{2}z_1(s)\Big)\,ds &= \int_0^t \Big(\tfrac{3}{2}(1+2s) - \tfrac{1}{2}\cdot 1\Big)\,ds \\ &= \int_0^t (1+3s)\,ds \\ &= t + \tfrac{3t^2}{2} \;=\; \tilde{g}_0(t).\checkmark \end{aligned}$$

Component 1.

$$\begin{aligned} \int_0^t \Big(-\tfrac{1}{2}z_0(s) + \tfrac{3}{2}z_1(s)\Big)\,ds &= \int_0^t \Big(-\tfrac{1}{2}(1+2s) + \tfrac{3}{2}\cdot 1\Big)\,ds \\ &= \int_0^t (1-s)\,ds \\ &= t - \tfrac{t^2}{2} \;=\; \tilde{g}_1(t).\checkmark \end{aligned}$$

Summary

$$\boxed{\widetilde{K} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}, \quad \widetilde{\mathbf{g}}(t) = \begin{pmatrix} t + \frac{3}{2}t^2 \\ t - \frac{1}{2}t^2 \end{pmatrix}, \quad \mathbf{z}(t) = \begin{pmatrix} 1+2t \\ 1 \end{pmatrix}.}$$

2 Type-2 Volterra Integral Equation (VIE-2)

The vector VIE-2 is

$$\mathbf{z}(t) = \widetilde{\mathbf{g}}(t) + \int_0^t \widetilde{K}(t-s)\,\mathbf{z}(s)\,ds, \qquad t \geq 0.$$

Diagonal system

Choose $K = \operatorname{diag}(1, 2)$ and exact solutions

$$y_1(t) = t, \qquad y_2(t) = t^2.$$

The forcing functions satisfy $y_r = g_r + \int_0^t k_r\, y_r(s)\, ds$:

$$g_1(t) = t - \int_0^t s\, ds = t - \tfrac{t^2}{2},$$
$$g_2(t) = t^2 - \int_0^t 2s^2\, ds = t^2 - \tfrac{2t^3}{3}.$$

Transformed system

$$\mathbf{z}(t) = P\mathbf{y}(t) = \begin{pmatrix} t + t^2 \\ t - t^2 \end{pmatrix}, \qquad \widetilde{K} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix},$$

$$\widetilde{\mathbf{g}}(t) = P\mathbf{g}(t) = \begin{pmatrix} \left(t - \frac{t^2}{2}\right) + \left(t^2 - \frac{2t^3}{3}\right) \\ \left(t - \frac{t^2}{2}\right) - \left(t^2 - \frac{2t^3}{3}\right) \end{pmatrix} = \begin{pmatrix} t + \frac{t^2}{2} - \frac{2t^3}{3} \\ t - \frac{3t^2}{2} + \frac{2t^3}{3} \end{pmatrix}.$$

Verification

Component 0.

$$\begin{aligned} \tilde{g}_0(t) + \int_0^t \left(\tfrac{3}{2} z_0(s) - \tfrac{1}{2} z_1(s)\right) ds &= \left(t + \tfrac{t^2}{2} - \tfrac{2t^3}{3}\right) + \int_0^t \left(\tfrac{3}{2}(s + s^2) - \tfrac{1}{2}(s - s^2)\right) ds \\ &= \left(t + \tfrac{t^2}{2} - \tfrac{2t^3}{3}\right) + \int_0^t (s + 2s^2)\, ds \\ &= \left(t + \tfrac{t^2}{2} - \tfrac{2t^3}{3}\right) + \left(\tfrac{t^2}{2} + \tfrac{2t^3}{3}\right) \\ &= t + t^2 \;=\; z_0(t). \checkmark \end{aligned}$$

Component 1.

$$\begin{aligned} \tilde{g}_1(t) + \int_0^t \left(-\tfrac{1}{2} z_0(s) + \tfrac{3}{2} z_1(s)\right) ds &= \left(t - \tfrac{3t^2}{2} + \tfrac{2t^3}{3}\right) + \int_0^t \left(-\tfrac{1}{2}(s + s^2) + \tfrac{3}{2}(s - s^2)\right) ds \\ &= \left(t - \tfrac{3t^2}{2} + \tfrac{2t^3}{3}\right) + \int_0^t (s - 2s^2)\, ds \\ &= \left(t - \tfrac{3t^2}{2} + \tfrac{2t^3}{3}\right) + \left(\tfrac{t^2}{2} - \tfrac{2t^3}{3}\right) \\ &= t - t^2 \;=\; z_1(t). \checkmark \end{aligned}$$

Summary

$$\boxed{\widetilde{K} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}, \quad \widetilde{\mathbf{g}}(t) = \begin{pmatrix} t + \frac{1}{2}t^2 - \frac{2}{3}t^3 \\ t - \frac{3}{2}t^2 + \frac{2}{3}t^3 \end{pmatrix}, \quad \mathbf{z}(t) = \begin{pmatrix} t + t^2 \\ t - t^2 \end{pmatrix}.}$$

3 Volterra Integro-Differential Equation (VIDE)

The vector VIDE is

$$\mathbf{z}'(t) = \widetilde{a}(t)\,\mathbf{z}(t) + \widetilde{\mathbf{g}}(t) + \int_0^t \widetilde{K}(t-s)\,\mathbf{z}(s)\,ds, \qquad \mathbf{z}(0) = \mathbf{z}_0.$$

Diagonal system

Choose $a = \mathrm{diag}(1,2)$, $K = \mathrm{diag}(1,2)$, and exact solutions

$$y_1(t) = t, \qquad y_2(t) = t^2, \qquad \mathbf{y}(0) = \mathbf{0}.$$

Substituting into $y_r' = a_r y_r + g_r + \int_0^t k_r y_r(s)\,ds$ and solving for g_r:

Component 1.

$$1 = 1\cdot t + g_1(t) + \int_0^t s\,ds = t + g_1(t) + \tfrac{t^2}{2} \implies g_1(t) = 1 - t - \tfrac{t^2}{2}.$$

Component 2.

$$2t = 2t^2 + g_2(t) + \int_0^t 2s^2\,ds = 2t^2 + g_2(t) + \tfrac{2t^3}{3} \implies g_2(t) = 2t - 2t^2 - \tfrac{2t^3}{3}.$$

Transformed system

Since $a = K = \mathrm{diag}(1,2)$, both transform to the same matrix:

$$\widetilde{a} = \widetilde{K} = PDP^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

The transformed solution and initial condition are

$$\mathbf{z}(t) = P\mathbf{y}(t) = \begin{pmatrix} t+t^2 \\ t-t^2 \end{pmatrix}, \qquad \mathbf{z}(0) = P\mathbf{0} = \mathbf{0}.$$

The transformed forcing is

$$\begin{aligned} \widetilde{\mathbf{g}}(t) = P\mathbf{g}(t) &= \begin{pmatrix} g_1(t) + g_2(t) \\ g_1(t) - g_2(t) \end{pmatrix} \\ &= \begin{pmatrix} \left(1 - t - \frac{t^2}{2}\right) + \left(2t - 2t^2 - \frac{2t^3}{3}\right) \\ \left(1 - t - \frac{t^2}{2}\right) - \left(2t - 2t^2 - \frac{2t^3}{3}\right) \end{pmatrix} = \begin{pmatrix} 1 + t - \frac{5t^2}{2} - \frac{2t^3}{3} \\ 1 - 3t + \frac{3t^2}{2} + \frac{2t^3}{3} \end{pmatrix}. \end{aligned}$$

Verification

We verify $\mathbf{z}'(t) = \widetilde{a}\,\mathbf{z}(t) + \widetilde{\mathbf{g}}(t) + \int_0^t \widetilde{K}(t-s)\,\mathbf{z}(s)\,ds$ for each component.

The integral term is the same as in the VIE-2 verification above (since $\widetilde{K}$ and $\mathbf{z}$ are identical):

$$\int_0^t \Big(\widetilde{K}\,\mathbf{z}(s)\Big)_0\,ds = \int_0^t (s + 2s^2)\,ds = \tfrac{t^2}{2} + \tfrac{2t^3}{3},$$

$$\int_0^t \Big(\widetilde{K}\,\mathbf{z}(s)\Big)_1\,ds = \int_0^t (s - 2s^2)\,ds = \tfrac{t^2}{2} - \tfrac{2t^3}{3}.$$

Component 0. Left-hand side: $z_0'(t) = 1 + 2t$.

Right-hand side:

$$\begin{aligned}
&\big(\widetilde{a}\,\mathbf{z}\big)_0 + \tilde{g}_0 + \int_0^t \big(\widetilde{K}\,\mathbf{z}(s)\big)_0\,ds \\
&\quad= \Big(\tfrac{3}{2}(t+t^2) - \tfrac{1}{2}(t-t^2)\Big) + \Big(1 + t - \tfrac{5t^2}{2} - \tfrac{2t^3}{3}\Big) + \Big(\tfrac{t^2}{2} + \tfrac{2t^3}{3}\Big) \\
&\quad= (t + 2t^2) + \Big(1 + t - \tfrac{5t^2}{2} - \tfrac{2t^3}{3}\Big) + \Big(\tfrac{t^2}{2} + \tfrac{2t^3}{3}\Big) \\
&\quad= 1 + 2t + \Big(2 - \tfrac{5}{2} + \tfrac{1}{2}\Big)t^2 + \Big(-\tfrac{2}{3} + \tfrac{2}{3}\Big)t^3 \\
&\quad= 1 + 2t.\checkmark
\end{aligned}$$

Component 1. Left-hand side: $z_1'(t) = 1 - 2t$.

Right-hand side:

$$\begin{aligned}
&\big(\widetilde{a}\,\mathbf{z}\big)_1 + \tilde{g}_1 + \int_0^t \big(\widetilde{K}\,\mathbf{z}(s)\big)_1\,ds \\
&\quad= \Big(-\tfrac{1}{2}(t+t^2) + \tfrac{3}{2}(t-t^2)\Big) + \Big(1 - 3t + \tfrac{3t^2}{2} + \tfrac{2t^3}{3}\Big) + \Big(\tfrac{t^2}{2} - \tfrac{2t^3}{3}\Big) \\
&\quad= (t - 2t^2) + \Big(1 - 3t + \tfrac{3t^2}{2} + \tfrac{2t^3}{3}\Big) + \Big(\tfrac{t^2}{2} - \tfrac{2t^3}{3}\Big) \\
&\quad= 1 + (1-3)t + \Big(-2 + \tfrac{3}{2} + \tfrac{1}{2}\Big)t^2 + \Big(\tfrac{2}{3} - \tfrac{2}{3}\Big)t^3 \\
&\quad= 1 - 2t.\checkmark
\end{aligned}$$

Summary

$$\boxed{\widetilde{a} = \widetilde{K} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}, \quad \widetilde{\mathbf{g}}(t) = \begin{pmatrix} 1 + t - \frac{5}{2}t^2 - \frac{2}{3}t^3 \\ 1 - 3t + \frac{3}{2}t^2 + \frac{2}{3}t^3 \end{pmatrix}, \quad \mathbf{z}(t) = \begin{pmatrix} t + t^2 \\ t - t^2 \end{pmatrix}, \quad \mathbf{z}(0) = \mathbf{0}.}$$