Analytic Solutions for Scalar Volterra Test Cases

The three scalar equation types under consideration are:

$$\text{VIE-1:}\quad g(t) = \int_0^t K(t-s)\, y(s)\, ds, \tag{1}$$

$$\text{VIE-2:}\quad y(t) = g(t) + \int_0^t K(t-s)\, y(s)\, ds, \tag{2}$$

$$\text{VIDE:}\quad y'(t) = a(t)\, y(t) + g(t) + \int_0^t K(t-s)\, y(s)\, ds, \quad y(0) = y_0. \tag{3}$$

In each case the kernel K is specified as a function of the lag $\tau = t - s \geq 0$, and the array `kernel_values` stores $K(0), K(\Delta t), K(2\Delta t), \ldots$.

The following lemma is used in two of the six examples.

Lemma 1.

$$\int_0^t e^s \sin s\, ds = \tfrac{1}{2}\big[e^s(\sin s - \cos s)\big]_0^t = \tfrac{1}{2}\big(e^t(\sin t - \cos t) + 1\big).$$

Consequently,

$$\int_0^t e^{-(t-s)} \sin s\, ds \;=\; e^{-t}\int_0^t e^s \sin s\, ds \;=\; \frac{\sin t - \cos t + e^{-t}}{2}.$$

Proof. Differentiate the anti-derivative directly: $\frac{d}{ds}\big[e^s(\sin s - \cos s)\big] = e^s(\sin s - \cos s) + e^s(\cos s + \sin s) = 2e^s \sin s$. Dividing by 2 gives the first identity. The second follows by writing $e^{-(t-s)} = e^{-t} \cdot e^s$ and pulling e^{-t} outside the integral. □

1 Type-1 Volterra Integral Equation (VIE-1)

1.1 Polynomial kernel: $K(\tau) = 2 + \tau$, $y(t) = t$

The convolution kernel is $K(t-s) = 2 + (t-s)$. We claim $y(t) = t$ and derive g by direct computation:

$$\begin{aligned} g(t) &= \int_0^t \big(2 + (t-s)\big) \cdot s\, ds = \int_0^t \big(2s + ts - s^2\big)\, ds \\ &= \Big[s^2 + \tfrac{t}{2}s^2 - \tfrac{s^3}{3}\Big]_0^t = t^2 + \tfrac{t^3}{2} - \tfrac{t^3}{3} = t^2 + \tfrac{t^3}{6}. \end{aligned}$$

Summary.

$$\boxed{K(\tau) = 2 + \tau, \quad g(t) = t^2 + \tfrac{t^3}{6}, \quad y(t) = t.}$$

1.2 Exponential kernel: $K(\tau) = e^{\tau}$, $y(t) = \cos t - \sin t$

The convolution kernel is $K(t-s) = e^{t-s}$. Differentiating the VIE-1 equation with respect to t gives a direct formula for y:

$$g'(t) = K(0)\,y(t) + \int_0^t \frac{\partial}{\partial t} e^{t-s}\,y(s)\,ds = y(t) + \int_0^t e^{t-s}\,y(s)\,ds = y(t) + g(t),$$

hence

$$y(t) = g'(t) - g(t).$$

With $g(t) = \sin t$ this yields $y(t) = \cos t - \sin t$.

Verification. Using the standard integrals $\int e^{-s}\cos s\,ds = \frac{1}{2}e^{-s}(\sin s - \cos s)$ and $\int e^{-s}\sin s\,ds = -\frac{1}{2}e^{-s}(\sin s + \cos s)$,

$$\begin{aligned}\int_0^t e^{t-s}(\cos s - \sin s)\,ds &= e^t \int_0^t e^{-s}(\cos s - \sin s)\,ds \\ &= e^t\Big[\tfrac{1}{2}e^{-s}(\sin s - \cos s) + \tfrac{1}{2}e^{-s}(\sin s + \cos s)\Big]_0^t \\ &= e^t\big[e^{-s}\sin s\big]_0^t = e^t\big(e^{-t}\sin t - 0\big) = \sin t \;=\; g(t).\checkmark\end{aligned}$$

Summary.

$$\boxed{K(\tau) = e^{\tau}, \quad g(t) = \sin t, \quad y(t) = \cos t - \sin t.}$$

2 Type-2 Volterra Integral Equation (VIE-2)

2.1 Decaying exponential kernel: $K(\tau) = e^{-\tau}$, $y(t) = \sin t$

The convolution kernel is $K(t-s) = e^{-(t-s)}$. We claim $y(t) = \sin t$ and derive g from the VIE-2 equation:

$$g(t) = y(t) - \int_0^t e^{-(t-s)}\sin s\,ds.$$

By Lemma **??**,

$$\int_0^t e^{-(t-s)}\sin s\,ds = \frac{\sin t - \cos t + e^{-t}}{2},$$

so

$$g(t) = \sin t - \frac{\sin t - \cos t + e^{-t}}{2} = \frac{\sin t + \cos t - e^{-t}}{2}.$$

Verification.

$$g(t) + \int_0^t e^{-(t-s)}\sin s\,ds = \frac{\sin t + \cos t - e^{-t}}{2} + \frac{\sin t - \cos t + e^{-t}}{2} = \sin t \;=\; y(t).\checkmark$$

Summary.

$$\boxed{K(\tau) = e^{-\tau}, \quad g(t) = \tfrac{1}{2}(\sin t + \cos t - e^{-t}), \quad y(t) = \sin t.}$$

2.2 Cosine kernel: $K(\tau) = 2\cos\tau$, $y(t) = e^t$

The convolution kernel is $K(t-s) = 2\cos(t-s)$. We claim $y(t) = e^t$ and compute the integral using the angle-sum formula $\cos(t-s) = \cos t \cos s + \sin t \sin s$:

$$\int_0^t 2\cos(t-s)\, e^s\, ds = 2\cos t \int_0^t e^s \cos s\, ds + 2\sin t \int_0^t e^s \sin s\, ds.$$

The two standard anti-derivatives are

$$\int e^s \cos s\, ds = \tfrac{1}{2} e^s(\cos s + \sin s),$$
$$\int e^s \sin s\, ds = \tfrac{1}{2} e^s(\sin s - \cos s),$$

which follow from the same integration-by-parts argument used in Lemma **??**. Evaluating from 0 to t:

$$\int_0^t e^s \cos s\, ds = \tfrac{1}{2}\big(e^t(\cos t + \sin t) - 1\big),$$
$$\int_0^t e^s \sin s\, ds = \tfrac{1}{2}\big(e^t(\sin t - \cos t) + 1\big).$$

Therefore,

$$\begin{aligned}\int_0^t 2\cos(t-s)\, e^s\, ds &= \cos t\big(e^t(\cos t + \sin t) - 1\big) + \sin t\big(e^t(\sin t - \cos t) + 1\big) \\ &= e^t \cos^2 t + e^t \sin t \cos t - \cos t + e^t \sin^2 t - e^t \sin t \cos t + \sin t \\ &= e^t(\cos^2 t + \sin^2 t) + \sin t - \cos t \\ &= e^t + \sin t - \cos t.\end{aligned}$$

Hence $g(t) = e^t - (e^t + \sin t - \cos t) = \cos t - \sin t$.

Verification.

$$g(t) + \int_0^t 2\cos(t-s)\, e^s\, ds = (\cos t - \sin t) + (e^t + \sin t - \cos t) = e^t \ = \ y(t).\checkmark$$

Summary.

$$\boxed{K(\tau) = 2\cos\tau, \quad g(t) = \cos t - \sin t, \quad y(t) = e^t.}$$

3 Volterra Integro-Differential Equation (VIDE)

3.1 Pure ODE: $K = 0$, $a(t) = -1$, $y(0) = 2$

With $K = 0$ the equation reduces to the first-order linear ODE

$$y'(t) = -y(t) + g(t), \qquad y(0) = 2.$$

Finding y. We take $g(t) = t$ and solve by the integrating factor $\mu(t) = e^t$:

$$\frac{d}{dt}(e^t y) = e^t(y' + y) = te^t.$$

Integrating by parts,

$$e^t y = \int t e^t \, dt = (t-1)e^t + C.$$

Applying $y(0) = 2$:

$$e^0 \cdot 2 = (0-1)e^0 + C \implies C = 3,$$

giving

$$y(t) = 3e^{-t} + t - 1.$$

Verification.

$$y'(t) = -3e^{-t} + 1, \qquad -y(t) + t = -(3e^{-t} + t - 1) + t = -3e^{-t} + 1.$$

So $y' = -y + t = g(t)$. ✓

Summary.

$$\boxed{K = 0, \quad a(t) = -1, \quad g(t) = t, \quad y(0) = 2, \quad y(t) = 3e^{-t} + t - 1.}$$

3.2 Full VIDE: $K(\tau) = e^{-\tau}$, $a(t) = \dfrac{1}{1+t^2}$, $y(0) = 0$

We claim $y(t) = \sin t$ and derive g from the VIDE equation:

$$g(t) = y'(t) - a(t)\, y(t) - \int_0^t K(t-s)\, y(s)\, ds.$$

Step 1: differentiate y. $y'(t) = \cos t$.

Step 2: $a(t)y(t)$.

$$a(t)\, y(t) = \frac{\sin t}{1+t^2}.$$

Step 3: the convolution integral. By Lemma **??** with $y(s) = \sin s$,

$$\int_0^t e^{-(t-s)} \sin s \, ds = \frac{\sin t - \cos t + e^{-t}}{2}.$$

Step 4: assemble g.

$$\begin{aligned} g(t) &= \cos t - \frac{\sin t}{1+t^2} - \frac{\sin t - \cos t + e^{-t}}{2} \\ &= \cos t - \frac{\sin t}{1+t^2} - \frac{\sin t}{2} + \frac{\cos t}{2} - \frac{e^{-t}}{2} \\ &= \frac{3\cos t}{2} - \frac{\sin t}{2} - \frac{e^{-t}}{2} - \frac{\sin t}{1+t^2}. \end{aligned}$$

Verification.

$$
\begin{aligned}
a(t)\,y(t) + g(t) + \int_0^t e^{-(t-s)} \sin s \, ds &= \frac{\sin t}{1+t^2} + \left(\frac{3\cos t}{2} - \frac{\sin t}{2} - \frac{e^{-t}}{2} - \frac{\sin t}{1+t^2} \right) + \frac{\sin t - \cos t + e^{-t}}{2} \\
&= \frac{3\cos t}{2} - \frac{\sin t}{2} - \frac{e^{-t}}{2} + \frac{\sin t - \cos t + e^{-t}}{2} \\
&= \frac{3\cos t}{2} - \frac{\cos t}{2} + \frac{-\sin t + \sin t}{2} + \frac{-e^{-t} + e^{-t}}{2} \\
&= \cos t \;=\; y'(t). \checkmark
\end{aligned}
$$

Summary.

$$
\boxed{K(\tau) = e^{-\tau}, \quad a(t) = \frac{1}{1+t^2}, \quad g(t) = \frac{3\cos t}{2} - \frac{\sin t}{2} - \frac{e^{-t}}{2} - \frac{\sin t}{1+t^2}, \quad y(0) = 0, \quad y(t) = \sin t.}
$$